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Yang-Baxter Equation in Long Range Interacting Systems

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Abstract

We consider the $su(p)$ spin chains with long range interactions and the spin generalization of the Calogero-Sutherland models. We show that their properties derive from a transfer matrix obeying the Yang-Baxter equation. We obtain the expression of the conserved quantities of the dynamical models and we diagonalise them. In the spin chain case, we establish the connection between the degeneracies of the spectrum and the representation theory of the Yangians. We use a correspondence with the dynamical models to diagonalise the Hamiltonian. Finally, we extend the previous results to the case of a trigonometric R matrix.

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1. INTRODUCTION.

The most remarkable properties of the XXX chain with long range interactions [1] [2] are that its spectrum is additive and that the states are created by filling a "Dirac sea" with particles obeying a "Generalised Pauli principle" [3]. Recently, it has become apparent [4] that the algebras underlying the symmetries of these models are the Yangians [5]. In [4], the first generators of the Yangian had been obtained, and the aim of this paper is to display the full algebra. To characterise it, we have constructed a transfer matrix which satisfies the exchange relations [6] [7] resulting from the Yang-Baxter [8] equation, (often called the $RLL = LLR$ relations). In the limit of infinite separation of the sites, this transfer matrix reduces to the usual XXX chain form [8]. To prove the exchange relations, we have used the differentials operators defined in [9] [10] [11] to "quantize" the spectral parameter of a standard transfer matrix.

In this work, we also consider models for which the lattice sites are replaced by dynamical particles, see also [12] [13] [14] [15]. They are the spin generalisations of the Calogero-Sutherland models [16] [17] [18]. In these models, there also exists a transfer matrix obeying the Yang-Baxter equation. However, an important difference between the two situations is that the transfer matrix of the dynamical models always commutes with the Hamiltonian, whereas in the lattice model case, it commutes only if the lattice is translation invariant. The generating function for the Hamiltonians is not given by the trace of the transfer matrix, because this trace does not commute with the Yangian. In the dynamical case it is given by the quantum determinant. In the lattice case, the determinant is a c-number which contains enough information to recover the spectrum degeneracies.

In section two we define the dynamical models. In part 2.1 we prove the Yang-Baxter equation for the transfer matrix. In part 2.2, we derive the conserved quantities which we diagonalise in part 2.3.

In section three, we consider the spin models. In part 3.1, we relate the decomposition of the $su(p)$-spin chain into irreducible Yangian representations to the spectrum degeneracies given in [4]. In part 3.2 and 3.3, we describe the relation between the spin chains and the dynamical models and we use this relation to find the energy spectrum and the highest weight vectors.

In section four, we show how the results of the previous sections can be extended to the case of a trigonometric R matrix.

2. THE DYNAMICAL MODELS.

The dynamical models are $su(p)$ generalisations of the Calogero-Sutherland model. There are M particles interacting by long range forces. Their positions are parameterised by complex numbers z_i, $i = 1, \cdots, M$, and each particle carries a spin σ with p possible values. If the particles are on the unit circle, we take $z_j = \exp(i\nu_j)$, and if they are on the line we take $z_j = \exp \nu_j$, with ν_j real. Their dynamics are governed by the Hamiltonian :

$$H_D = \sum_{j=1}^{M} (z_j \partial_{z_j})^2 + \sum_{i \neq j} \lambda (P_{ij} + \lambda) \frac{z_i z_j}{(z_i - z_j)(z_j - z_i)} \tag{2.1}$$

where λ is a coupling constant and P_{ij} exchanges the spins of the particles i and j.

Integrability is guaranteed by the existence of a Lax pair. It consists of two matrices L_{ij} and M_{ij} with operator entries obeying:

$$\Big[\; H_D \; , \; L_{ij} \; \Big] = \sum_k \Big(L_{ik} M_{kj} \; - \; M_{ik} L_{kj} \Big) \tag{2.2}$$

A possible choice is given by:

$$\begin{aligned} L_{ij} &= \delta_{ij} z_j \partial_{z_j} + (1 - \delta_{ij}) \lambda \theta_{ij} P_{ij} \\ M_{ij} &= -\delta_{ij} 2\lambda \sum_k (h_{ik} P_{ik}) + 2\lambda (1 - \delta_{ij}) h_{ij} P_{ij} \end{aligned} \tag{2.3}$$

with:

$$\theta_{ij} = \frac{z_i}{z_i - z_j} \quad , \quad h_{ij} = \theta_{ij} \theta_{ji} \tag{2.4}$$

We denote by X_j^{ab}, $a, b = 1, \cdots, p$, the matrices which act as $|a\rangle\langle b|$ on the spins of the j^{th} particle and leave the other particles untouched. Using equation (2.2) and the fact that $\left[H_D, X_j^{ab} \right] = \sum_k X_k^{ab} M_{kj}$ and $\sum_j M_{ij} = 0$, one deduces that the quantities defined by:

$$T_n^{ab} \; = \; \sum_{ij} X_i^{ab} (L^n)_{ij} \tag{2.5}$$

commute with the Hamiltonian H_D. Here L^n denotes the n^{th} power of the matrix L_{ij}. Since the T_n^{ab} do not commute with each other, the spectrum is degenerate. We study their algebra in the next section.

2.1. The transfer matrices.

In order to arrange the algebra of the T_n^{ab}'s, we introduce the transfer matrix $T(u)$ obeying the Yang-Baxter equation:

$$R_{00'}(u-v)\ T^0(u)\ T^{0'}(v)\ =\ T^{0'}(v)\ T^0(u)\ R_{00'}(u-v) \tag{2.6}$$

According to standard notations [7], $T^0(u)$ denotes the matrix $T(u) \otimes 1$ and $T^{0'}(v)$ the matrix $1 \otimes T(v)$. The matrix $R(u)$ is the solution of the Yang-Baxter equation given by:

$$R(u)\ =\ u + \lambda P_{00'} \tag{2.7}$$

where $P_{00'}$ is the permutation operator which exchanges the two auxiliary spaces 0 and $0'$. Equation (2.6) expresses the non-commutativity of the operator matrix elements of $T(u)$. The expression which we obtain for $T^0(u)$ is given by:

$$T^0(u)\ =\ 1 + \lambda \sum_{i,j=1}^{M} P_{0i} \left(\frac{1}{u-L} \right)_{ij} \tag{2.8}$$

where L is the matrix defined in eq. (2.3). If we set $T^0(u) = \sum_{a,b=1}^{p} X_{ba}^0 T^{ab}(u)$, and expand it in powers of $\frac{1}{u}$, we find:

$$T^{ab}(u)\ =\ \delta^{ab} + \sum_{n=0}^{\infty} \frac{\lambda}{u^{n+1}} T_n^{ab} \tag{2.9}$$

where the T_n^{ab}'s have been defined in eq. (2.5). We can motivate this expression as follows: It commutes with the Hamiltonian H_D, the $\frac{1}{u^2}$ coefficient coincides with (a slight generalization of) the generators of the Yangians identified in [4], and, as we will discuss in a next section, in a specific limit it gives back the transfer matrix of an XXX chain. When the z_i which define the functions θ_{ij} in equation (2.4) are complex numbers of modulus one, an important property of this transfer matrix is its hermiticity: $T^{ab\dagger}(u) = T^{ba}(\overline{u})$.

In order to prove (2.6), we use operators introduced in [10][12][11] in the physics literature and known in mathematics under the name of Dunkl operators [9][19] [20] [21] [22] [23]. Let us define the three permutation groups: Σ_1, Σ_2 and Σ_3 respectively generated by K_{ij}, P_{ij} and the product $(P_{ij}K_{ij})$. The operator K_{ij} exchanges the positions z_i, and

P_{ij} exchanges the spins at positions i and j: i.e. $K_{ij}z_j = z_iK_{ij}$ and $P_{ij}\sigma_j = \sigma_iP_{ij}$. We define the differentials (Dunkl operators):

$$D_i = z_i\partial_{z_i} + \lambda\sum_{i\neq j}\theta_{ij}K_{ij} \tag{2.10}$$

They obey the relations:

$$\begin{aligned} K_{ij}\ D_i &= D_j\ K_{ij} \\ K_{ij}\ D_j &= D_i\ K_{ij} \\ K_{ij}\ D_k &= D_k\ K_{ij} \quad \text{if } k\neq i,j \\ \Big[\ D_i\ ,\ D_j\ \Big] &= \lambda(D_i - D_j)\ K_{ij} \end{aligned} \tag{2.11}$$

We also define a projection π which consists to replace the permutation K_{ii+1} by the permutation P_{ii+1} after it has been moved to the right of an expression. Equivalently, we set equal to 1 a permutation of Σ_3 appearing to the right of an expression. We use it to eliminate the permutations of Σ_1 by replacing them with those of Σ_2. For example:

$$\begin{aligned} \pi(K_{12}) &= P_{12} \\ \pi(K_{12}K_{23}) &= P_{23}P_{12} \end{aligned} \tag{2.12}$$

One can view this projection as the result of acting on bosonic wave functions symmetric under permutations in Σ_3. The expression (2.8) of the transfer matrix then takes the following form [2]:

$$T^0(u) \ = \ \pi\Big(1 + \lambda\sum_{i=1}^{M}\frac{P_{0i}}{u-D_i}\Big) \tag{2.13}$$

which results from $\pi(D_i^n) = \sum_j L_{ij}^n$. Since $T^0(u)$ is invariant under simultaneous permutations of the spins and the coordinates, a product of projections can be replaced by the projection of the product. Therefore, we omit the symbol π and set equal to one any permutation of Σ_3 appearing to the right of an expression. Equation (2.6) then recasts into the form :

$$\begin{aligned} &(u-v+\lambda P_{00'})\Big(\sum_{i=1}^{M}(1+\frac{\lambda P_{0i}}{u-D_i})(1+\frac{\lambda P_{0'i}}{v-D_i}) + \sum_{i\neq j}(\frac{\lambda P_{0i}}{u-D_i})(\frac{\lambda P_{0'j}}{v-D_j})\Big) \\ =&\Big(\sum_{i=1}^{M}(1+\frac{\lambda P_{0'i}}{v-D_i})(1+\frac{\lambda P_{0i}}{u-D_i}) + \sum_{i\neq j}(\frac{\lambda P_{0'j}}{v-D_j})(\frac{\lambda P_{0i}}{u-D_i})\Big)(u-v+\lambda P_{00'}) \end{aligned} \tag{2.14}$$

[2] A similar expression has independently be obtained in [24]

First consider the sum over i: each term in the sum is a Yang-Baxter equation for an elementary transfer matrix, $T_i(u) = 1 + \frac{\lambda P_{oi}}{u - D_i}$, with a spectral parameter equal to $u - D_i$. The equality is therefore satisfied by each term independently. Then, consider the sum with $i \neq j$. Using the identities:

$$\begin{aligned} P_{00'}P_{0i}P_{0'j} &= K_{ij}P_{0i}P_{0'j}(K_{ij}P_{ij}) \\ P_{0i}P_{0'j}P_{00'} &= K_{ij}P_{0'i}P_{0j}(K_{ij}P_{ij}) \end{aligned} \tag{2.15}$$

the remaining terms reduce to :

$$(u - v + \lambda K_{ij}) \left(\frac{1}{v - D_i}\right) \left(\frac{1}{u - D_j}\right) = \left(\frac{1}{u - D_j}\right) \left(\frac{1}{v - D_i}\right) (u - v + \lambda K_{ij}) \tag{2.16}$$

or equivalently,

$$(u - v + \lambda K_{ij})(u - D_j)(v - D_i) = (v - D_i)(u - D_j)(u - v + \lambda K_{ij}) \tag{2.17}$$

This results from eq. (2.11). It proves the Yang-Baxter equation (2.6) for the transfer matrix given in eq. (2.8).

We now give an alternative description of the transfer matrix. Let us modify the differentials D_i introduced in (2.10) as follows:

$$\begin{aligned} \widehat{D}_i &= D_i - \lambda \sum_{j<i} K_{ij} \\ &= z_i \partial_{z_i} + \lambda \sum_{j>i} \theta_{ij} K_{ij} - \lambda \sum_{j<i} \theta_{ji} K_{ij} \end{aligned} \tag{2.18}$$

Unlike the D_i, the differentials $\widehat{D}_i$ commute but they do not transform covariantly under the permutations any more. They obey the defining relations of a degenerate affine Hecke Algebra [25]:

$$\begin{aligned} &\left[\widehat{D}_i \ , \ \widehat{D}_j \right] = 0 \\ &\left[K_{ii+1} \ , \widehat{D}_k \right] = 0 \quad \text{if } k \neq i, i+1 \\ &K_{ii+1}\widehat{D}_i - \widehat{D}_{i+1}K_{ii+1} = \lambda \end{aligned} \tag{2.19}$$

From these relations, one deduces the commutation relations:

$$\left[K_{ii+1} \ , \ (u - \widehat{D}_i)(u - \widehat{D}_{i+1}) \right] = 0 \tag{2.20}$$

which imply that:

$$\widehat{\Delta}_M(u) = \prod_{i=1}^{M}(u - \widehat{D}_i) \tag{2.21}$$

is invariant under the permutations of the coordinates (Σ_1).

In term of these differentials, we can define a transfer matrix which satisfies the Yang-Baxter equation. It is given by:

$$\widehat{T}^0(u) = \Big(1 + \frac{\lambda P_{01}}{u - \widehat{D}_1}\Big)\Big(1 + \frac{\lambda P_{02}}{u - \widehat{D}_2}\Big)\cdots\Big(1 + \frac{\lambda P_{0M}}{u - \widehat{D}_M}\Big) \tag{2.22}$$

The right hand side of (2.22) is the co-product of elementary transfer matrices $\widehat{T}_i$ with their spectral parameters equal to $u - \widehat{D}_i$.

The transfer matrix $T^0(u) = \pi(\widehat{T}^0(u))$ will satisfy the Yang-Baxter equation if we can replace the projection of the product $\widehat{T}^0\widehat{T}^{0'}$ by the product of the projections. For this to be true, $\widehat{T}^0$ applied on a bosonic wave function must still be a bosonic wave function. Equivalently, we must have:

$$\pi(\Sigma_3\widehat{T}^0) = \pi(\widehat{T}^0) \tag{2.23}$$

Since the denominator $\widehat{\Delta}_M(u)$ of $\widehat{T}^0(u)$ commutes with Σ_1, it suffices to show the above equality for the product $\widehat{\Delta}_M(u)\widehat{T}^0(u)$. Then, (2.23) results from the identities:

$$\pi\big(K_{ii+1}(u-\widehat{D}_i+P_{0i})(u-\widehat{D}_{i+1}+P_{0i+1})\big) = P_{ii+1}\pi\big((u-\widehat{D}_i+P_{0i})(u-\widehat{D}_{i+1}+P_{0i+1})\big) \tag{2.24}$$

which can be shown using (2.19).

For a small number of particles, we have verified that $\pi(\widehat{T}^0)$ coincides with the expression of T^0 in (2.13) and we shall assume it is true in the following.

2.2. The conserved quantities.

One usually generates the conserved quantities by means of the trace of the transfer matrix. This defines quantities which commute among themselves but not with the T_n^{ab}. Here, in order to satisfy this condition, we are led to take the quantum determinant as the generating function. It is given by [26] [6][25]:

$$Det_q\ T(u) \ = \ \sum_{\sigma\in\Sigma_p} \epsilon(\sigma)T_{1\sigma_1}(u-(p-1)\lambda)T_{2\sigma_2}(u-(p-2)\lambda)\cdots T_{p\sigma_p}(u) \tag{2.25}$$

The first two non-trivial terms produce the momentum and the Hamiltonian:

$$\begin{aligned} Det_q T(u) \;=\; & 1 + \frac{\lambda}{u} M + \frac{\lambda}{u^2}\Big(\sum_{j=1}^{M} z_j \partial_{z_j} + \lambda \frac{M(M-1)}{2}\Big) \\ & + \frac{\lambda}{u^3}\Big(H_D + \lambda(M-1)\sum_{j=1}^{M} z_j \partial_{z_j} + \lambda^2 \frac{M(M-1)(M-2)}{6}\Big) + \cdots \end{aligned} \tag{2.26}$$

To compute the quantum determinant of $T(u)$ it easier first to evaluate the determinant of the unprojected transfer matrix (2.22). It is equal to the product of the quantum determinants of each factor $\widehat{T}_i$ defining $\widehat{T}(u)$ and is given by:

$$Det_q \widehat{T}(u) = \frac{\widehat{\Delta}_M(u+\lambda)}{\widehat{\Delta}_M(u)} \tag{2.27}$$

where $\widehat{\Delta}_M(u)$ is defined in (2.21). Then, arguing as in the last section, we can replace $\widehat{\Delta}_M(u)$ by its projection $\Delta_M(u)$ in (2.27) to obtain the quantum determinant of $T(u)$.

The coefficients C_p of the polynomial $\Delta_M(u)$ define a complete set of commuting operators:

$$C_p = \sum_{i_1 < \cdots < i_p} \pi(\widehat{D}_{i_1} \widehat{D}_{i_2} \cdots \widehat{D}_{i_p}) \tag{2.28}$$

The first three coefficients are given by:

$$\begin{aligned} C_1 &= \sum_i z_i \partial_{z_i} \\ C_2 &= \sum_{i<j} z_i \partial_{z_i} z_j \partial_{z_j} + \lambda(\lambda + P_{ij}) \frac{z_i z_j}{(z_i - z_j)^2} \\ C_3 &= \sum_{i<j<k} z_i \partial_{z_i} z_j \partial_{z_j} z_k \partial_{z_k} + \lambda^2 \frac{z_i z_j z_k}{(z_i - z_j)(z_j - z_k)(z_k - z_i)} (P_{ij} P_{jk} - P_{jk} P_{ij}) \\ &\quad + \Big(\frac{z_i z_j}{(z_i - z_j)^2} \lambda(\lambda + P_{ij}) z_k \partial_{z_k} \quad + \text{cyclic permutations}\Big) \end{aligned} \tag{2.29}$$

When there is no spin dependence (Calogero-Sutherland model), we have verified that the expression for $\Delta_M(u)$ is equal to the determinant defined in [27]. However, the evaluation of the determinant [27] is much easier than the one of $\Delta(u)$ and shows that the coupling constant λ enters the conserved quantities via the combination $\lambda(\lambda+1)$.

Finally, let us discuss the rational limit of the preceeding results. If we set $z_j = \exp(\gamma\nu_j)$ and let γ go to zero, the Hamiltonian becomes:

$$H_D = -\sum_{j=1}^{M}(\partial_{\nu_j})^2 + \sum_{i\neq j}\frac{\lambda(P_{ij}+\lambda)}{(\nu_i-\nu_j)^2} \tag{2.30}$$

In order to reach this limit, we need to rescale D_j and λ by γ. When γ goes to zero, the T_n^{ab} algebra degenerates to:

$$\left[T_n^{ab}, T_m^{cd}\right] = \delta_{bc}T_{n+m}^{ad} - \delta_{ad}T_{n+m}^{cb} \tag{2.31}$$

In that case, the traces $\sum_a T_n^{aa}$ define the conserved quantities which commute with the T_n^{ab}.

2.3. Diagonalisation of the conserved quantities

Let us obtain here the eigenvalues of $\Delta(u)$. Our method is first to diagonalise the differentials $\widehat{D}_i$. In this way we find the eigenvectors of $\widehat{\Delta}(u)$. Since $\widehat{\Delta}(u)$ is invariant under Σ_3, we can symmetrise these eigenvectors with respect to the spin and coordinate permutations to obtain the eigenvectors of $\Delta(u)$ with the same eigenvalue. In order to do this, it is convenient to make a gauge transformation which amounts to substitute $z_i\partial_{z_i} - \frac{\lambda}{2}\sum_{j\neq i}\frac{z_i+z_j}{z_i-z_j}$ for $z_i\partial_{z_i}$ everywhere. Hereafter, we indicate with a prime the gauged transformed quantities. The gauge transformed differentials $\widehat{D}_i'$ take the following form:

$$\widehat{D}_i' = z_i\partial_{z_i} + \lambda\left(\frac{M+1}{2} - i\right) + \lambda\sum_{j>i}\theta_{ij}(K_{ij}-1) - \lambda\sum_{j<i}\theta_{ji}(K_{ij}-1) \tag{2.32}$$

Their action leaves the space of polynomials in z_i invariant. We consider the basis of monomials:

$$\Phi_{[n]}(z_1, z_2, \cdots, z_M) = z_1^{n_1}z_2^{n_2}\cdots z_M^{n_M} \tag{2.33}$$

where $[n]$ is a sequence of positive integers. To a sequence $[n]$, we associate the partition $|n|$ where we arrange the n_k in decreasing order. We define an order on the partitions by saying that $|n|$ is larger than $|n'|$ if the first non vanishing difference $n_k - n_k'$ is positive. It follows from the same argument as in [17] that the differentials $\widehat{D}_i'$ are represented by block triangular matrices in the basis $\Phi_{[n]}(z_k)$. Namely:

$$\widehat{D}_i'\Phi_{[n]}(z_k) = \sum_{n'}(d_i)_{[n'][n]}\Phi_{[n']}(z_k) \tag{2.34}$$

with $(d_i)_{[n'][n]} = 0$ if $|n'|$ is larger than $|n|$. Therefore, the eigenvalues of the differentials $\widehat{D}'_i$ are given by the eigenvalues of the block matrices on the diagonal. Let us consider such a block: $d_i^{|n|} = \left((d_i)_{[n'][n'']}\right)$ with $|n|$ a fixed partition $|n| = (n_1 \geq n_2 \geq \cdots \geq n_M)$. In the basis $|n_{\sigma_1}, n_{\sigma_2}, \cdots, n_{\sigma_M}>$, the $d_i^{|n|}$ take the following form:

$$d_i^{|n|}|n_{\sigma_1}, \cdots, n_{\sigma_M}>= \left(n_{\sigma_i} + \lambda(\frac{M+1}{2} - i + \sum_{j>i} X_{ij} - \sum_{j<i} X_{ji})\right)|n_{\sigma_1}, \cdots, n_{\sigma_M}> \quad (2.35)$$

where the X_{ij} are defined by:

$$X_{ij}|\cdots, n_{\sigma_i}, \cdots, n_{\sigma_j}, \cdots>= \begin{cases} -|\cdots, n_{\sigma_i}, \cdots, n_{\sigma_j}, \cdots> & \text{if } n_{\sigma_i} > n_{\sigma_j} \\ 0 & \text{if } n_{\sigma_i} = n_{\sigma_j} \\ |\cdots, n_{\sigma_j}, \cdots, n_{\sigma_i}, \cdots> & \text{if } n_{\sigma_i} < n_{\sigma_j} \end{cases} \quad (2.36)$$

The matrices $d_i^{|n|}$ are triangular when we order the states inside a block by saying that $[n']$ is larger than $[n'']$ if the last non vanishing difference $n'_k - n''_k$ is positive. With the global order induced by this choice the matrices $\widehat{D}'_i$ are also triangular. It follows that the eigenvalues, $\delta_i^{[n]}$, of $d_i^{|n|}$ are the diagonal matrix elements, $(d_i)_{[n][n]}$. One readily sees that the multiplets of eigenvalues, $(\delta_i^{[n]})_{i=1,M}$, of the $d_i^{|n|}$'s are all obtained by permuting the components of the multiplet:

$$(\delta_i^{|n|}) = \left(n_i + \lambda(i - \frac{M+1}{2})\right) \quad (2.37)$$

As a result, the corresponding eigenvectors of $\widehat{D}'_i$ form a degenerate set of eigenvectors of $\widehat{\Delta}'(u)$ with the eigenvalue:

$$\delta^{|n|}(u) = \prod_{j=1}^{M} \left(u - n_j - \lambda(j - \frac{M+1}{2})\right) \quad (2.38)$$

These eigenvectors form a representation of Σ_1, isomorphic to the obvious representation of the permutations on the sequences $[n]$. To obtain the eigenvectors $\Phi(z_i, \sigma_i)$ of $\Delta'(u)$, one must combine these eigenvectors with a spin component and symmetrise the tensor product with respect to Σ_3. The eigenvectors of $\Delta(u)$ are finally obtained when we multiply $\Phi(z_i, \sigma_i)$ by the function of z_i which removes the gauge transformation. The eigenfunctions write:

$$\Psi(z_i, \sigma_i) = \Phi(z_i, \sigma_i) \prod_{i<j} (z_i - z_j)^{-\lambda} (z_1 \cdots z_M)^{\frac{(M-1)\lambda}{2}} \quad (2.39)$$

There is an alternative (fermionic) construction of the transfer matrix (2.8) if we modify the definition of π and define $\pi_F(K_{12}) = -P_{12}$ instead of (2.12). In this case the sign of λ has to be changed in the definitions of D_i (2.10),(2.18),(2.32) and, consequently in the expression of the eigenvalues of $\Delta(u)$ (2.38). The fermionic eigenstates of $\Delta(u)$ take the following form:

$$\Psi(z_i, \sigma_i) = \Phi(z_i, \sigma_i) \prod_{i<j} (z_i - z_j)^{\lambda} (z_1 \cdots z_M)^{-\frac{(M-1)\lambda}{2}} \tag{2.40}$$

Where $\Phi(z_i, \sigma_i)$ is obtained by antisymetrising the eigenstates of $\widehat{D}'_i$ with respect to Σ_3. This construction will be useful to establish an equivalence between the dynamical and spin models in section 3.

2.4. A remark on the shift operators.

We recall that the Calogero-Sutherland Hamiltonian is given by (2.1) where P_{ij} has been set equal to one :

$$H_\lambda = \sum_{j=1}^{M} (z_j \partial_{z_j})^2 + \lambda(\lambda+1) \sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)(z_j - z_i)} \tag{2.41}$$

The shift operators [19][9][20][28], which we denote by Λ_B and Λ_F, intertwine the Calogero-Sutherland Hamiltonians with coupling constants λ and $(\lambda \pm 1)$. They are characterised by :

$$\begin{aligned} \Lambda_F \ H_{\lambda-1} &= H_\lambda \ \Lambda_F \\ \Lambda_B \ H_\lambda &= H_{\lambda-1} \ \Lambda_B \end{aligned} \tag{2.42}$$

Hence, if ϕ is an eigenfunction of $H_{\lambda-1}$, then $\Lambda_F \phi$ is an eigenfunction of H_λ, and similarly with Λ_B. These intertwiners can therfore be used to compute the eigenfunctions of the Calogero-Sutherland Hamiltonian with an integer coupling constant λ using those of the free Hamiltonian H_0. They can be understood from the Yangian point of view. Indeed, notice that in the representation where the permutation operators P_{ij} are equal to ± 1 the dynamical Hamiltonian (2.1) reduces to the Calogero-Sutherland Hamiltonian with coupling constant λ and $(\lambda - 1)$.

To construct the shift operators, let us define bosonic and fermionic projections π_B and π_F by $\pi_B(K_{ij}) = +1$ and $\pi_F(K_{ij}) = -1$ respectively when this permutation is at the right of an expression. The quantities $\pi_B\left(\widehat{\Delta}(u)\right) = \Delta_\lambda(u)$ and $\pi_F\left(\widehat{\Delta}(u)\right) = \Delta_{-\lambda}(u) =$

$\Delta_{\lambda-1}(u)$ are the generating functions for the commuting Hamiltonians of the Calogero-Sutherland models with coupling constants respectively equal to λ and $(\lambda-1)$. The equality $\Delta_{-\lambda}(u) = \Delta_{\lambda-1}(u)$ is due to the fact that $\Delta_\lambda(u)$ only depends on $\lambda(\lambda+1)$. Consider now the operator $\widehat{\Lambda}$ defined by :

$$\widehat{\Lambda} = \sum_\sigma \epsilon(\sigma) D^{M-1}_{\sigma_{M-1}} \cdots D^2_{\sigma_2} D_{\sigma_1} \tag{2.43}$$

where the sum is over the permutations. Since the differentials D_i transform covariantly under the permutations and commute with $\widehat{\Delta}(u)$, $\widehat{\Lambda}$ commutes with $\widehat{\Delta}(u)$ and is odd under the permutations :

$$\begin{aligned} \widehat{\Lambda}\ \widehat{\Delta}(u) &= \widehat{\Delta}(u)\ \widehat{\Lambda} \\ K_{ij}\ \widehat{\Lambda} &= -\widehat{\Lambda}\ K_{ij} \end{aligned} \tag{2.44}$$

Projecting the first of these equations with π_B or π_F gives :

$$\begin{aligned} \Lambda_B\ \Delta_\lambda(u) &= \pi_B(\widehat{\Lambda}\ \widehat{\Delta}(u)) = \pi_B(\widehat{\Delta}(u)\ \widehat{\Lambda}) = \Delta_{\lambda-1}(u)\ \Lambda_B \\ \Lambda_F\ \Delta_{\lambda-1}(u) &= \pi_F(\widehat{\Lambda}\ \widehat{\Delta}(u)) = \pi_F(\widehat{\Delta}(u)\ \widehat{\Lambda}) = \Delta_\lambda(u)\ \Lambda_F \end{aligned} \tag{2.45}$$

Here we have defined $\Lambda_B = \pi_B(\widehat{\Lambda})$ and $\Lambda_F = \pi_F(\widehat{\Lambda})$ and used the fact that $\widehat{\Delta}(u)$ commutes with K_{ij}. It shows that Λ_B, Λ_F are shift operators.

The shift operators for two particles are :

$$\Lambda_{B,F} = \pi_{B,F}(D_1 - D_2) = z_1\partial_{z_1} - z_2\partial_{z_2} \pm \lambda\frac{z_1+z_2}{z_1-z_2} \tag{2.46}$$

Finally, we point out that the Calogero-Sutherland Hamiltonian has a symmetry when $\lambda \to \lambda^{-1}$ which is discussed in [29].

3. THE SPIN MODELS.

The Hamiltonian of the $su(p)$ spin model is [1][2]:

$$H_S = \sum_{i\neq j} \frac{\widetilde{z}_i\widetilde{z}_j}{(\widetilde{z}_i - \widetilde{z}_j)(\widetilde{z}_j - \widetilde{z}_i)}(P_{ij} - 1) \tag{3.1}$$

The indices i, j refer to the sites of the chain which we take of length N and the $\widetilde{z}_k$ are complex numbers parametrising the positions of the sites. The trigonometric spin chain corresponds to $\widetilde{z}_k = \exp(i2\pi k/N)$.

Unlike in the usual cases, this model is not solvable by the Bethe ansatz. To solve it, we first decompose the spin chain Hilbert space into multiplets of degenerate eigenstates. Then we compute the energy spectrum by acting with the Hamiltonian on a particular state (e.g. the highest weight vector) in each multiplet. These states are identifed by using a relation bewteen the spin and the dynamical models.

3.1. The transfer matrix and the degeneracies.

In this section, we use the transfer matrix formalism to deduce the decomposition of the $su(p)$ spin chain into irreducible Yangian representations. The transfer matrix is the limit $\lambda \to \infty$, $\frac{u}{\lambda} = x$ fixed, of the matrix (2.8):

$$T^0(x) = 1 + \sum_{i,j=1}^{N} P_{0i} \Big(\frac{1}{x - L} \Big)_{ij} \tag{3.2}$$

with $L_{ij} = (1 - \delta_{ij}) \widetilde{\theta}_{ij} P_{ij}$. $\widetilde{\theta}_{ij}$ is defined as θ_{ij} in eq.(2.4) with $\widetilde{z}_i$ substituted for z_i. If we put the sites on the real axis and let the distance between them go to infinity with $0 < \widetilde{z}_1 << \widetilde{z}_2 \cdots << \widetilde{z}_N$, the $\widetilde{\theta}_{ij}$ converge to the step function $\theta(i - j)$ and the transfer matrix reduces to its usual form:

$$T^0(x) = \Big(1 + \frac{P_{01}}{x} \Big) \Big(1 + \frac{P_{02}}{x} \Big) \cdots \Big(1 + \frac{P_{0N}}{x} \Big) \tag{3.3}$$

For generic values of the complex numbers $\widetilde{z}_j$, the representation of the Yangian algebra $Y(sl_p)$ obtained in this way is irreducible. Thus, its quantum determinant is a c-number which we can evaluate on any vector. Choosing the vector with all spins σ_j equal to p gives:

$$Det_q \; T(x) \;\; = \;\; 1 + \sum_{i,j=1}^{N} \Big(\frac{1}{x - \Theta} \Big)_{ij} = \;\; \frac{\Delta_N(x+1)}{\Delta_N(x)} \tag{3.4}$$

Here Θ is the $N \times N$ matrix with matrix elements $\widetilde{\theta}_{ij}$ and $\Delta_N(x)$ is its characteristic polynomial:

$$\Delta_N(x) = det(x - \Theta) \tag{3.5}$$

Although the transfer matrix satisfies the exchange relations (2.6) for all values of the parameters $\widetilde{z}_i$, it commutes with the Hamiltonian (3.1) only if [4]:

$$\sum_j \widetilde{\theta}_{ij} \widetilde{\theta}_{ji} (\widetilde{\theta}_{ij} - \widetilde{\theta}_{ji}) = 0 \tag{3.6}$$

In order to satisfy this condition, we choose either $\widetilde{z}_k = \exp(i 2\pi k/N)$ with $k = 1, \cdots, N$, (trigonometric models), or, $\widetilde{z}_k = \exp(\gamma k)$, with k integers, (hyperbolic models). Diagonalising Θ, we obtain the following expression for Δ_N in the trigonometric case:

$$\Delta_N(x) = \prod_{j=1}^{N} \big(x - \frac{N+1}{2} + j \big) \tag{3.7}$$

In the trigonometric case, as a result of the Hermiticity of the transfer matrix, the Yangian algebra is completely reducible. Let us here deduce which possible irreducible representations can occur in the decomposition of the spin chain Hilbert space. Irreducible representations of the $su(p)$ Yangians are characterised by $p-1$ polynomials $P_k(x)$, $k = 1, \cdots, p-1$ [25][30] in the following way: in a canonical normalisation, the transfer matrix, $T_c(x)$, is a rational function of x and one can find a highest weight state, $|\Lambda>$, such that:

$$T_c(x)|\Lambda>= \begin{pmatrix} 1 & * & \cdots & * \\ 0 & \frac{P_{p-1}(x+1)}{P_{p-1}(x)} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{P_1 \ldots P_{p-1}(x+1)}{P_1 \ldots P_{p-1}(x)} \end{pmatrix} |\Lambda> \tag{3.8}$$

In each irreducible block appearing in the decomposition, the transfer matrix applied on the highest weight differs from a canonical form (3.8) by a multiplicative factor $\phi(x)$. Using the fact that $\Delta(x)T(x)$ is polynomial in x and that the roots of $\Delta(x)$ form a string of consecutive integers, one can show that $\phi(x)$ is necessarilly of the form: $\phi(x) = \frac{P_p(x+1)}{P_p(x)}$ where $P_p(x)$ is a polynomial in x. If we evaluate the quantum determinant of $T(x)$ in this block we obtain the following equation for $\Delta(x)$:

$$\begin{aligned} \Delta(x) \; &= P_p(x)P_p(x-1)\cdots P_p(x-p+1) \\ &\times P_{p-1}(x)P_{p-1}(x-1)\cdots P_{p-1}(x-p+2)\cdots \\ &\vdots \\ &\times P_1(x) \end{aligned} \tag{3.9}$$

The possible solutions for the polynomials $P_k(x)$ thus correspond to the possible ways to partition the roots of $\Delta(x)$, $\{l_j = j - \frac{N+1}{2}, j = 1, \cdots, N\}$, into sets of k-strings $\{\gamma_k - n, n = 0, \cdots, k-1\}$ where γ_k are the roots of $P_k(x)$. Each distinct solution for the set of polynomials $P_k(x)$ corresponds to a highest weight state.

The classification of the invariant subspaces, each with a unique highest weight state is now straightforward, and reproduces the empirically-determined structure obtained by analysis of numerical diagonalisation studies of H [4]. The partition into strings can be described by a sequence of $N-1$ binary digits "1" or "0" inserted between consecutive roots, indicating whether consecutive roots belong to the same string or not. A "0" can be added to each end of the sequence, so "0" indicates the end of a string. The vacuum state, which only occurs if N is a multiple of p, is obtained when all the roots are partitioned into

consecutive p-strings, so $P_k(x) = 1$ for $k < p$. This corresponds to the "trivial"(singlet) representation; for $p = 3$ and $N = 9$, the binary sequence is "0110110110", or in the motif picture [4] (11)(11)(11).

Each motif corresponds to a set of $su(p)$ sub-representations which give the degeneracies of the spectrum. In the $su(2)$ case for example, the series of q consecutive "0" bordered by "1"s correspond to the natural spin $\left(\frac{q-1}{2}\right)$ representaion of $su(2)$. The representation content of a motif is the tensor product of these representations. In the $su(p)$ case, the description of the the motifs in terms of representations is more involved. However, the multiplicities can be simply obtained by considering the $q = 0$ limit of the trigonometric models considered in part 4 (cristal basis).

3.2. The transfer matrix and the spectrum.

The previous classification characterises the possible invariant subspaces without telling which irreducible blocks appear and with which multiplicities. Neither does it give the energies of the motifs. To answer these questions we establish a relation between the spin and the dynamical models. More precisely, we characterise a class of states $|\Psi\rangle$ which are annihilated by the last line of the transfer matrix (except the diagonal element), $T^{pa}|\Psi\rangle = 0$ for $a = 1, \cdots, p-1$, and on which the first $(p-1)$ lines and columns of (3.2) act as the dynamical transfer matrix (2.8). Using this connection we show that any motif is actually realised on the spin chain and we compute its energy.

To specify this class of states, it is useful to describe the spin chain Hilbert space in terms of magnon states. A M-magnon states $|\Psi\rangle$ is given by:

$$|\Psi\rangle = \sum_{\alpha_1,\cdots,\alpha_M} \sum_{0\le n_1<\cdots<n_M<N} \psi^{\alpha_1,\cdots,\alpha_M}_{n_1,\cdots,n_M} \sigma^{\alpha_1}_{n-1} \cdots \sigma^{\alpha_M}_{n_M} |\Omega\rangle \tag{3.10}$$

where σ^α_n, with $\alpha = 1, \cdots, p-1$, denote the matrices $X^{\alpha p}_n = |\alpha\rangle\langle p|$ acting on the n^{th} spin, and $|\Omega\rangle$ is the ferromagnetic state with all spin equal to p: $|\Omega\rangle = |p, \cdots, p\rangle$. Let us extend the range of definition of $\psi^{\alpha_1,\cdots,\alpha_M}_{n_1,\cdots,n_M}$ to $0 \le n_i < N$ by requiring that it is symmetric under the simultanous permutations of the coordinates n_i and the spins α_i and that it vanishes when at least two of the n_i are equal. To a M-magnon state we associate the polynomial $\Psi(z_i; \alpha_i)$ of degree less than $(N-1)$ in each variable z_i such that :

$$\Psi(\omega^{n_1}, \cdots, \omega^{n_M}; \alpha_i) = \psi^{\alpha_1,\cdots,\alpha_M}_{n_1,\cdots,n_M} \tag{3.11}$$

These polynomials are symmetric under the simultanueous permutations of the spins and coordinates (Σ_3). The class of states we consider are associated to polynomials of the form :

$$\Psi(z_i,\alpha_i)=\prod_{p=1}^{M} z_p \prod_{p<q}(z_p-z_q)\ \Phi(z_i,\alpha_i) \tag{3.12}$$

with $\Phi(z_i,\alpha_i)$ a polynomial of degree less than $N-M$ in each variable z_i, antisymmetric with respect to Σ_3.

In the next section, we show that on the class of states specified by Eq. (3.12), the spin transfer matrix acts as the matrix $t^{a\,b}(x)$ given by:

$$t^{a\,b}(x)\Psi(z_i;\alpha_i)=\left(1+\sum_{s=1}^{M}\frac{X_s^{a\,b}}{x-D_s}\right)\Psi(z_i;\alpha_i)\quad ;\ a,b=1,\cdots,p-1 \tag{3.13}$$

where,

$$D_s=z_s\partial_{z_s}-\frac{N+1}{2}+\sum_{r\neq s}\theta_{sr}K_{sr} \tag{3.14}$$

The number of particles is the number of magnons. Furthermore, these states are annihilated by the elements $T^{p\,a}(x)$, $a=1,\cdots,p-1$ of the transfer matrix.

Before we establish the equality of $T^{a\,b}$ and $t^{a\,b}$, let us show how it enables to associate an invariant subspace of $T(x)$ to an invariant subspace of $t(x)$. For this, let us diagonalise the product $\delta(x)=\prod_{s=1}^{M}(x-\widehat{D}_s)$ where $\widehat{D}_s=D_s-\sum_{r<s}K_{rs}$. By the fermionic construction of section 2.3 it is achieved by diagonalising the $\widehat{D}_s$ on polynomials of the form:

$$\Psi(z_i)=\prod_{p=1}^{M} z_p \prod_{p<q}(z_p-z_q)\ \Phi(z_i) \tag{3.15}$$

The eigenvalues of $\delta(x)$ are characterised by a partition $|n|=(n_1\geq n_2\geq\cdots\geq n_M)$ and are equal to:

$$\delta_{|n|}=\prod_{s=1}^{M}(x-f_s)\quad ;\ \text{with } f_s=n_s+M+1-s-\frac{N+1}{2} \tag{3.16}$$

The corresponding eigenstates of $\delta(x)$ are such that the monomial of highest degree of $\Phi(z_i)$ is equal to $z_{\sigma_1}^{n_1}\cdots z_{\sigma_M}^{n_M}$. They are associated to a state of the chain by the above correspondance only if $n_s\leq N-M-1$, therfore the f_s must satisfy the following inequalities :

$$-\frac{N-1}{2}\leq f_M<\cdots<f_2<f_1\leq\frac{N-1}{2}-1 \tag{3.17}$$

In addition, at most $(p-1)$ integers n_s can be equal, i.e. at most $(p-1)$ integers f_s can be consecutive, since the spin wave function $\Phi(z_i, \alpha_i)$ (3.12) is antisymmetric.

When we evaluate the quantum determinant of $T(x)$ on this class of states, we obtain the following equation for $\Delta(x)$:

$$\Delta(x) = P_1(x) \cdots P_p(x)\ \delta_{|n|}(x-1) \tag{3.18}$$

where the polynomals $P_k(x)$ have been defined in (3.8). Comparing this equation with (3.9), we see that the f_s's completely determine the position of the roots of the polynomials $P_k(x)$. Namely, the way to recover the binary sequence which characterises the invariant subspace consists in inserting a "1" between consecutive roots of $\Delta(x)$ if the smallest root is equal to f_i for some i, and a "0" between all other consecutive roots.

This result extends to the $su(p)$ transfer matrix the known correspondence between the $su(2)$ spin chain and the Calogero-Sutherland Hamiltonians [1][2][31] [32]. One can show that on the states (3.12) the spin chain Hamiltonian acts as follows:

$$H = \sum_{s=1}^{M} (\widehat{D}_s + \frac{1}{2} + \frac{N}{2})(\widehat{D}_s + \frac{1}{2} - \frac{N}{2}) \tag{3.19}$$

Therefore, the energy of a motif f_s is:

$$H = \sum_{s=1}^{M}\ \epsilon(f_s) \quad ;\ \text{with } \epsilon(f_s) = \left(f_s + \frac{1}{2}\right)^2 - \frac{N^2}{4} \tag{3.20}$$

All states in the same invariant subspace are degenerate. The energy spectrum (3.20) satisfies an additivity property: The symbols "1" of a motif code the occupation numbers of quasi-particles with energies $\epsilon(f_s)$. These quasiparticles obey a generalised exclusion principle [3] which is reflected in the rules to fill a motif.

3.3. The correspondence between the spin and dynamical models.

To establish the correspondence between the chain and the dynamical models, it is convenient to use the expression (2.13) of the transfer matrix before it is projected by π. Here, we consider the additive form (2.13), a similar argument can also be applied to the multiplicative form (2.22). Let us enlarge the Hilbert space of the chain by introducing states $|\Gamma\rangle$ which are described by a spin and coordinate wave function as follows:

$$|\Gamma\rangle = \sum_{\alpha_i} \sum_{n_i} \Gamma^{\alpha_1,\cdots,\alpha_N}_{n_1,\cdots,n_N} |\alpha_i\rangle \otimes |n\rangle \tag{3.21}$$

where $|\alpha_i\rangle$ describe the spin configurations of the chain and the permutation:

$$n = \begin{pmatrix} 1 & 2 & \dots & N \\ n_1 & n_2 & \dots & n_N \end{pmatrix} \tag{3.22}$$

describes the coordinate configurations. On this Hilbert space, there is a natural representation of the operators z_i and K_{ij} given by:

$$\begin{aligned} z_k|n\rangle &= \widetilde{z}_{n_k}|n\rangle \\ K_{ij}|n\rangle &= |(ij)n\rangle \end{aligned} \tag{3.23}$$

In this way, we obtain a realisation of the differentials $\mathcal{D}_i$ defined by:

$$\mathcal{D}_i = \sum_{j \neq i} \theta_{ij} K_{ij} \tag{3.24}$$

which obey the relations (2.11) with $\lambda = 1$. The Hilbert space of the chain is recovered when we restrict ourselves to (boson)states invariant under the simultanous permutations of the spins and the coordinates (Σ_3). Since the states of the chain are characterised by their spin wave function $\Gamma^{\alpha_1,\cdots,\alpha_N}_{1,2,\cdots,N}$, we can represent them by projecting them on the identity permutation:

$$\langle Id|\Gamma_B\rangle = \sum_{\alpha_i} \Gamma^{\alpha_1,\cdots,\alpha_N}_{1,2,\cdots,N} |\alpha_1\alpha_2\cdots\alpha_N\rangle \tag{3.25}$$

The projection by π of an operator X which preserves the bosons is defined by:

$$\langle Id|X|\Gamma_B\rangle = \pi(X)\langle Id|\Gamma_B\rangle \tag{3.26}$$

Thus, if we apply the projection π to the operator:

$$T(x) = 1 + \sum_{i=1}^{N} \frac{P_{0i}}{x - \mathcal{D}_i} \tag{3.27}$$

we recover the expression (3.2) of the transfer matrix.

For our purpose, it is more convenient to represent the magnon states by wave functions such that the magnons occupy the first M sites of the chain and the $N - M$ last spins are all equal to p. Therefore, we define a projection P_M acting in the M magnon sector by:

$$P_M|\alpha_1\cdots\alpha_N\rangle = \Big(\prod_{i=M+1}^{N} \delta_{\alpha_i,p}\Big)\ |\alpha_1\cdots\alpha_M\rangle \tag{3.28}$$

The M-magnon states which are characterised by the wave function (3.10), are given by the projection by P_M of the bosonic Hilbert space on the states:

$$|\Psi_M\rangle = \sum_{\alpha_i} \sum_{n_i} \psi^{\alpha_1,\cdots,\alpha_M}_{n_1,\cdots,n_M} |\alpha_1 \cdots \alpha_M\rangle \otimes |n_1 \cdots n_M\rangle \tag{3.29}$$

Here, because of the bosonic nature of the states, we need only keep the M first indices n_i of the permutation n to define the state $|n\rangle$. In the same way as in (3.26), we define the projection p_M of an operator X acting in the M-magnon sector by:

$$P_M X |\Gamma_B\rangle = p_M(X) P_M |\Gamma_B\rangle \tag{3.30}$$

Let us consider the projection by p_M of the first $p-1$ rows and columns of the transfer matrix (3.27):

$$p_M(T^{ab}) = p_M(\delta_{ab} + \sum_{i=1}^{N} \frac{X_i^{ab}}{x - \mathcal{D}_i}) \tag{3.31}$$

for $1 \leq a, b \leq p-1$.

First, only the first M first terms of the sum contribute because $p_M(X_i^{ab}) = 0$ if $i \geq M+1$. Then consider the representation of the differentials $\mathcal{D}_i$ on the projected Hilbert space:

$$\langle n|\mathcal{D}_1|\Psi_M\rangle = \sum_{i=1}^{M} \widetilde{\theta}_{n_1 n_i} K_{1i} \Psi_M(n_1, \cdots, n_M) + \sum_{k \neq n_1, \cdots, n_M} \widetilde{\theta}_{n_1 k} \Psi_M(k, n_2, \cdots, n_M) \tag{3.32}$$

The second sum can be extended to all k since we assume that the wave function vanishes when two arguments are equal. We now on specialise to $\widetilde{z}_k = \exp(i2\pi k/N)$. In that case, one can show that polynomials $P(z)$ of degree between 1 and $N-1$ satisfy the identity:

$$\left(z\partial_z - \frac{N+1}{2}\right) P(\widetilde{z}_k) = \sum_{l \neq k} \widetilde{\theta}_{kl}\ P(\widetilde{z}_l) \tag{3.33}$$

Therefore, on the class of wave functions associated to the polynomials (3.12) we can replace the differentials $\mathcal{D}_i$ by the differentials D_i defined in (3.14). This shows the equality of the transfer matrices $p_M(T^{ab})$ and t^{ab} defined in (3.13) on this class of states. Similarly, one could show that the matrix elements $p_M(T^{ap})$ vanish on those states which establishes the correspondence used in the last section.

Finally, consider the generalised Hamiltonian:

$$\widehat{H} = \sum_{i<j} \theta_{ij}\theta_{ji}K_{ij} \tag{3.34}$$

whose projection by π is the spin chain Hamiltonian. One can show [33] that it commutes with K_{ij} and $\mathcal{D}_i$ in the representation (3.23) if the conditions (3.6) are satisfied. Therefore, it defines a conserved quantity for the chain. Similarly as above, one can evaluate $p_M(\widehat{H})$ to obtain:

$$p_M(\widehat{H}) = \sum_{s=1}^{M}(\widehat{D}_s + \frac{1}{2} + \frac{N}{2})(\widehat{D}_s + \frac{1}{2} - \frac{N}{2}) \tag{3.35}$$

We used this equation to find the spectrum in the previous section. In the chain case, unlike in the dynamical case, we have not obtained the generating function for the conserved quantities.

4. TRIGONOMETRIC MODELS.

The aim of this section is to repeat the construction of the transfer matrix in the trigonometric case where the Hecke algebra plays the role of the permutation algebra. Consider the realisation of Yang's relations:

$$Y_{12}(u_1-u_2)Y_{23}(u_1-u_3)Y_{12}(u_2-u_3) = Y_{23}(u_2-u_3)Y_{12}(u_1-u_3)Y_{23}(u_1-u_2) \tag{4.1}$$

given by [34]:

$$Y_{ii+1}(u) = q^{2u}t_{ii+1} - t_{ii+1}^{-1} \tag{4.2}$$

where the t_{ii+1} are the generators of a Hecke algebra:

$$\begin{aligned} &(t_{ii+1}-q)(t_{ii+1}+q^{-1}) = 0 \\ &t_{ii+1}t_{ii+2}t_{ii+1} = t_{ii+2}t_{ii+1}t_{ii+2} \\ &[t_{ii+1}, t_{jj+1}] = 0 \quad \text{if } |i-j| \geq 2 \end{aligned} \tag{4.3}$$

If we let the permutations P_{ij} act in the natural way on the indices k, l of t_{kl}, the operators defined by:

$$R_{ij}(u_i-u_j) = Y_{ij}(u_i-u_j)P_{ij} \tag{4.4}$$

obey the Yang-Baxter equation:

$$R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3) = R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2) \tag{4.5}$$

A transfer matrix $\widehat{T}^0(u)$ which obeys the quadratic relations (2.6) is given by:

$$\widehat{T}^0(u) = L_1(u)L_2(u)\cdots L_M(u) \tag{4.6}$$

with:

$$L_i(u) = \frac{(q^{2u}y_i t_{0i} - t_{0i}^{-1})P_{0i}}{q^{-1}(q^{2u}y_i - 1)} \tag{4.7}$$

The y_i are spectral parameters and therefore commute with everything. As in section 2, the denominator of $L_i(u)$ is a normalisation factor which is not relevant for the discussion except to normalise the quantum determinant of the transfer matrix. Let us assume that there exists another realisation of the Hecke algebra (4.3) generated by the g_{ii+1} which commute with the t_{kk+1} and the P_{ij} but not with the y_j. We define q-Bosons (Fermions) states $|\Psi_B>, (|\Psi_F>)$ by:

$$\begin{aligned}(g_{ii+1} - t_{ii+1})|\Psi_B> &= 0\\ (g_{ii+1} + t_{ii+1}^{-1})|\Psi_F> &= 0\end{aligned} \tag{4.8}$$

We require the q-Bosonic subspace to be preserved by $\widehat{T}^0(u)$. For this, we define a projection π which substitutes t_{ii+1} for g_{ii+1} to the right of an expression. The analogous condition to (2.23) which ensures that $T^0 = \pi(\widehat{T}^0)$ preserves the Bosonic subspace and therefore obeys the Yang-Baxter equation is now:

$$\pi(g_{ii+1}\widehat{T}^0) = t_{ii+1}\pi(\widehat{T}^0) \tag{4.9}$$

It is realised if the y_i obey the relations of a affine Hecke algebra (see for example [35] for more information on Hecke algebras):

$$\begin{aligned}&[y_i, y_j] = 0\\ &[g_{ii+1}, y_j] = 0 \quad \text{if } j \neq i, i+1\\ &g_{ii+1}y_i - y_{i+1}g_{ii+1}^{-1} = 0\end{aligned} \tag{4.10}$$

and results from the equalities:

$$\pi(g_{ii+1}L_iL_{i+1}) = t_{ii+1}\pi(L_iL_{i+1}) \tag{4.11}$$

To obtain a representation of the transfer matrix we need to have the t_{ii+1} and the P_{ij} act on the spins and the g_{ii+1} act on the coordinates z_j of the wave function. The P_{ij} permute the spins, and for the t_{ii+1} we can use the standard representation of the Hecke

algebra on the spin chain (see for example [36]). For the g_{ii+1} we can use the following representation [37]:

$$g_{ii+1} = qK_{ii+1} - (q - q^{-1})\theta_{ii+1}(K_{ii+1} - 1) \tag{4.12}$$

Where θ_{ij} is defined as before eq.(2.4) and K_{ij} permutes the coordinates z_i and z_j of the wave-function. The y_j are given by:

$$y_j = g_{jj+1}^{-1} K_{jj+1} \cdots g_{jM}^{-1} K_{jM} . S_j . K_{1j} g_{1j} \cdots K_{j-1j} g_{j-1j} \tag{4.13}$$

where the operator S_k multiplies the coordinate z_k by a constant ρ:

$$S_k \Psi(z_1, \cdots, z_k, \cdots, z_M) = \Psi(z_1, \cdots, \rho z_k, \cdots, z_M) \tag{4.14}$$

The transfer matrix $\pi(\widehat{T}^0)$ is finally obtained when we use the bosonic condition (4.8) and the explicit form (4.12) of g_{ii+1} to eliminate the permutations K_{ii+1} to the right of an expression:

$$K_{ii+1} \to Y'_{ii+1}(\frac{z_i}{z_{i+1}}) = \frac{z_{i+1} t_{ii+1} - z_i t_{ii+1}^{-1}}{z_{i+1} q - z_i q^{-1}} \tag{4.15}$$

Note that the $Y'_{ii+1}(z)$ form a representation of Yang's algebra (4.1) for which the spectral parameters u_i are replaced by the coordinates z_i.

The quantum determinant of $T(u)$ is given by:

$$Det_q T(u) = \frac{\Delta_M(u+1)}{\Delta_M(u)} \tag{4.16}$$

where:

$$\Delta_M(u) = \pi\big((q^{2u} y_1 - 1) \cdots (q^{2u} y_M - 1)\big) \tag{4.17}$$

is the generating function for the conserved quantities. We do not repeat the diagonalisation of $\Delta(u)$ which is parallel to the discussion of the last section and yields the q-analogue of eq.(2.38).

The transfer matrix of the spin chain is obtained by setting $\rho = 1$ in the previous transfer matrix. As for the Haldane-Shastry chain, we have not obtained the the conserved quantities which commute with this transfer matrix.

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